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Theory of Linearized Magneto-Fluid-Mechanics

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The steady two-dimensional flow of a compressible, conducting fluid past a wavy wall or a thin symmetrical airfoil is studied for the case where the undisturbed, uniform magnetic field is oriented either perpendicular or parallel to the undisturbed uniform stream. Exact solutions of the linearized magnetogasdynamic equations are obtained for arbitrary values of Mach number *M*, Alfven number *A*, and magnetic Reynolds number for *Rm* for wavy walls of various forms. The solutions for a thin symmetrical airfoil are obtained in terms of their Fourier transforms, the inversions of thich are carried out only for the case of a perfectly conducting fluid. Included in the solutions are the effects on the flow fields of a system of magnetic dipoles and currents placed in the bodies(airfoils and wavy walls). For sub-fast-wave flow with infinite *Rm* and crossed undisturbed fields, the present theory gives a negative drag for certain current systems in the body. The slow and fast waves of magnetogasdynamics at infinite *Rm* are generalized to damped waves for arbitrary *Rm*. They become Mach waves in the limits $Rm \to 0$ or $A \to \infty$. When the undisturbed fields *Vo* and *Bo* are aligned, the upstream facing wave and a magnetic wake (or magnetic boundary layer) facing either downstream ur upstream are recognized in the compressible flows past wavy walls for large *Rm*. A reversed flow theorem for the upstream wake and the current-free classical ideal flows are developed for arbitrary *Rm*.

Lastly, the flow past wavy wall with finite thickness is obtained where the fluids above and below the wall may be different, i. e. the the free streams directions Mach Number, and magnetic Reynolds numbers, etc. for the flows above and below the wall may be different.

INTRODUCTION

Magnetogasdynamic flows are governed by higher order differential equations than ordinary gasdynamic flows giving rise to two wave systems where the magnetic Reynolds number $R_m=\infty$ instead of the single Mach wave system which characterizes the flow in ordinary gasdynamics, where $R_m=0$. An extensive literature (see Sears 1961) has accumulated on this subject when $R_m=0$ (ordinary gasdynamics) and $R_m=\infty$ (magnetogasdynamics with perfectly conducting fluids). Two of the most interesting phenomena of magnetogasdynamics in the case of aligned undisturbed fields are (i) the upstreamfacing wave which occurs when $M^2+A^2>1$, $M<1$, $A<1$, $R_m=\infty$ and (ii) the magnetic boundary layer (or magnetic wake) when $R_m\cong\infty$ which faces upstream for $M^2+A^2<1$ and downstream for $M^2+A^2>1$ (Resler and McCune 1960). Here A is the Alfven number and M is the Mach number. However, since previous investigations for arbitrary R_m have been limited to incompressible flows ($M=0$), it has not been shown that these controversial large R_m solutions characterize flows which occur under more general conditions and which reduce to ordinary subsonic and supersonic flows in the limit $R_m \to 0$.

In this paper, exact solutions of MHD flows past thin bodies with crossed and aligned undisturbed fields are studied for the entire ranges of A, M, and R_m. The following simplifying assumptions are made:

(a) The geometry of the boundaries (thin symmetrical airfoils and wavy walls) and the magnetic singularities (currents and magnetic dipoles) are such that the perturbations of the flow fields are small compared to free-stream values and vanish at infinity;

(b) The flow is steady and two-dimensional and the unperturbed magnetic and velocity fields lie in the plane of the flow perpendicular or aligned to each other with no current in the unperturbed stream;

(c) The fluid has constant properties, zero viscosity, and a scalar electrical conductivity;

(d) The flow process is either isothermal or approximately isentropic and the fluid is a perfect gas;

(e) The bodies (whose magnetic permeability μ_* may be different from that of the fluid, μ) are perfect insulators having imbedded currents or magnetic dipoles;

(f) There are no discontinuities in the flow fields (velocity, magnetic field, and pressure) except across the body surface.

As R_m changes from 0 to ∞, the solutions continuously change from the ordinary gasdynamic solutions to the infinite R_m MHD solutions, displaying damped waves for finite R_m. The drag curves of Resler and McCune (1959) and the flow fields of Broer and van Wijngaarden (1963) can be reproduced as the case $M=0$ of the present theory. The solutions for infinite R_m are also presented and used to demonstrate a negative drag. A reversed flow theorem for the upstream wake of incompressible sub-Alfvenic ($A<1$) flow past a magnetized body is obtained in a manner similar to that of Hasimoto (1959) who considered viscous flow with $R_m=\infty$. The current-free classicl ideal flow in MHD observed first by Sears and Resler (1959) for $R_m=\infty$ with aligned fields, and by Lary (1962) for finite R_m with aligned fields is reproduced here for both aligned and crossed fields. The present theory reduces to the ordinary gasdynamic of subsonic and supersonic flows in the limit of zero applied magnetic field ($A \to \infty$) as well as in the limit of zero conductivity of the fluid ($R_m \to 0$).

BASIC EQUATIONS

The fundamental equations governing MHD flows under the assumptions stated above are:

$$\nabla \times B = \mu J, \tag{1}$$

$$\nabla \cdot B = 0, \tag{2}$$

$$\nabla \times E = 0, \tag{3}$$

$$J = \sigma (E + V \times B), \tag{4}$$

$$\nabla \cdot (\rho V) = 0, \tag{5}$$

$$\rho (V \cdot \nabla) V = -\nabla P + J \times B, \tag{6}$$

$$P - P_0 = a_0^2 (\rho - \rho^0) \tag{7}$$

where B=magnetic induction, μ=magnetic permeability, J=current density, E=electric field, σ=electrical conductivity, V=flow velocity, ρ=density, P=pressure, and a_0 is the isothermal or isentropic sound speed at free stream conditions [depending on the assumption (d)].

For sufficiently small disturbances the state variables can be written in the form

$$\left.\begin{aligned} &V\equiv(1+\epsilon U_x,\ \epsilon U_y,\ 0),\\ &B\equiv B(\cos\theta+\epsilon b_x,\quad \sin\theta+\epsilon b_y,\ 0),\\ &\epsilon s\equiv\frac{\rho-\rho_0}{\rho_0},\ \epsilon p\equiv\frac{P-P_0}{\rho_0 U^2}=\frac{\epsilon s}{M^2},\\ &J\equiv\frac{B}{\mu L}(0,\ 0,\ \epsilon j),\end{aligned}\right\} \quad (8)$$

where U, B, ρ_0, P_0 are the magnitudes of the undisturbed state variables, $M=U/a_0$ is the Mach number, L is the wavelength or chord of the body, and θ is the angle between the free stream direction and the applied magnetic field. We use the dimensionless right-handed Cartesian coordinates x, y, and z. From Eqs. (1), (4), (8) and the assumption (b), only the z-components of E and J are not identically zero. Due to Eq. (3) and assumption (b), E is a constant: $E=-UB\hat{z}\sin\theta$. We denote by ϵ the thickness ratio of the obstacle or the amplitude to wavelength ratio of the wavy wall. We confine our attention to situations where $\epsilon\ll 1$ and each nondimensional state variable and its derivatives are of order ϵ. We define the magnetic vector potential $X(x, y)$ and the stream function $\Psi(x, y)$ such that

$$\frac{\partial X}{\partial x}\equiv -b,\qquad \frac{\partial X}{\partial y}\equiv b_x;\ \text{and}$$

$$\frac{\partial\Psi}{\partial x}\equiv -u_y,\qquad \frac{\partial\Psi}{\partial y}\equiv u_y+s. \quad (9)$$

The nondimensional current $j(x, y)$ and vorticity $w(x, y)$ can be written as

$$j=\frac{\partial b_y}{\partial x}-\frac{\partial b_x}{\partial y}=-\nabla^2 X,$$

$$w\equiv\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}=-\nabla^2\Psi+M^2\frac{\partial p}{\partial y} \quad (9)'$$

The introduction of Eqs. (8) and (9) into Eqs. (1)−(7), neglecting terms of higher order in ϵ, gives

$$\begin{pmatrix} \frac{1}{R_m}\nabla^2-\frac{\partial}{\partial x}, & \cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}, & -M^2\sin\theta\\ \frac{\sin\theta}{A^2}\nabla^2, & -\frac{\partial^2}{\partial x\partial y}, & (M^2-1)\frac{\partial}{\partial x}\\ \frac{\cos\theta}{A^2}\nabla^2, & -\frac{\partial^2}{\partial x^2}, & \frac{\partial}{\partial y}\end{pmatrix}\begin{pmatrix} X(x, y)\\ \Psi(x, y)\\ p(x, y)\end{pmatrix} \quad \begin{matrix}(10)\\(11)\\(12)\end{matrix}$$

where the magnetic Reynolds number $R_m=UL\sigma\mu$, and the Alfen number $A=\sqrt{U\rho_0\mu}/B$. Eliminating any two of the three quantities, Ψ, X, and p from Eqs. (10)—(12), we obtain the 5th order partial differenial equation

$$\Big[\frac{\partial}{\partial x}\Big(\frac{1}{K}\nabla^2-A^2\frac{\partial}{\partial x}\Big)\Big\{\nabla^2-M^2\frac{\partial^2}{\partial x^2}\Big\} +\nabla^2\Big\{\Big(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\Big)^2 -M^2\frac{\partial^2}{\partial x^2}\Big\}\Big](\Psi,\ x,\ p,)=0. \quad (13)$$

When $\theta=\frac{\pi}{2}$ (crossed fields), this reduces to

$$\Big[\frac{1}{K}\frac{\partial}{\partial x}\nabla^2\Big(\nabla^2-M^2\frac{\partial^2}{\partial x^2}\Big) +\Big(C_1^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\Big)\Big(C_2^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\Big)\Big](\Psi,\ x,\ p)=0 \quad (14)$$

where

$$C_1^2=\frac{1}{2}(A^2+M^2-1-C^2)\geq -1,$$

$$C_2^2=\frac{1}{2}(A^2+M^2-1+C^2)\geq 0,$$

$$C^2=\sqrt{\{1+(M+A^2)\{1+(M-A)^2\}}>1,$$

$$K=\frac{R_m}{A_2}. \quad (15)$$

When $\theta=0$ (aligned fields), Eq. (13) becomes

$$\Big[\frac{1}{K}\frac{\partial}{\partial x}\nabla\Big(\nabla^2-M^2\frac{\partial^2}{\partial x^2}\Big) +(1-M^2-A^2)\frac{\partial^2}{\partial x^2}\Big\{\frac{(1-M^2)(1-A^2)}{1-M^2-A^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big\}\Big](\Psi,\ x,\ p,)=0. \quad (16)$$

When $\mu=0$ (incompressible flow), Eq. (16) reduces to

$$\frac{\partial}{\partial x}\left\{\nabla^2-K(A^2-1)\frac{\partial}{\partial x}\right\}\nabla^2(\Psi, x, p)=0. \tag{17}$$

The relations between w and j, obtained from Eqs. (9), (11), and (12), are

$$\frac{\partial w}{\partial x}=\frac{1}{A^2}\left(\frac{\partial j}{\partial y}\sin\theta+\frac{\partial j}{\partial x}\cos\theta\right) \tag{18}$$

$$w=j\frac{M^2}{A^2}\cos\theta-\left(\nabla^2-M^2\frac{\partial x}{\partial x^2}\right)\Psi. \tag{19}$$

Since the system is dissipative, the perturbations must vanish at infinity. The application of the boundary conditions on the surface of the body, at $y=\pm 0$, is justified by the usual arguments of thin airfoil theory, i. e., the space-gradients of the flow variables are of order ϵ, and the surface of the body lies within a distance of order ϵ from $y=0$, while the magnetic field in the body is harmonic. Hence, the body may be replaced by distributions of singularities (sources, sinks, currents, and magnetic dipoles) along $y=0$. The total flux from the magnetic dipole distribution is necessarily zero. In this paper, we shall confine our attention to flows for which the net current in the body ($\iint j\,dxdy$ over the body cross section) is zero. Thus, the boundary conditions for a thin symmetrical airfoil can be written in the form

$$\left.\begin{aligned}
&\Psi(x,\pm 0)=\pm f(x),\\
&X(x,+0)-X(x,-0)=-m(x),\\
&\frac{\partial X(x,+0)}{\partial y}-\frac{\partial X(x,-0)}{\partial y}\\
&\qquad=-h(x),\\
&f(a)=f(b)=0,\\
&m(a)=m(b)=0,\\
&\int_b^a h(x)\,dx=0,
\end{aligned}\right\} \tag{20}$$

where $m(x)$ and $h(x)$ represent the distributions of magnetic poles and currents in the airfoil, and a and b are the x-coordinates of the leading and trailing edges of the airfoil.

Since the magnetic permeability μw of the airfoil does not appear in Eqs. (10)—(12) and (20), the solution for the flow past a thin airfoil is independent of μw.

GENERAL SOLUTIONS

We shall study first the flow past a thin symmetrical airfoil, and will then apply the results to the wavy wall problem. A flow field will be called symmetrical if each flow quantity is either an even or an odd function of y. A study of Eqs. (10)—(12) and Eq. (20) shows that symmetrical solutions for the flow past a thin symmetrical body do not exist unless $\theta=0$, or $\theta=\frac{\pi}{2}$. For these two cases, symmetrical solutions can be obtained in the form:

(i) for crossed fields

$$\left\{\begin{aligned}
&\Psi(x,-y)=-\Psi(x,y),\\
&\qquad p(x,-y)=p(x,y),\\
&X(x,-y)=X(x,y),\\
&\qquad m(x)=0;
\end{aligned}\right. \tag{21a}$$

(ii) for aligned fields

$$\left\{\begin{aligned}
&\Psi(x,-y)=-\Psi(x,y),\\
&\qquad p(x,-y)=p(x,y),\\
&X(x,-y)=-X(x,y),\\
&\qquad h(x)=0.
\end{aligned}\right. \tag{21b}$$

We define the Fourier transforms

$$\bar{F}(x,\eta)=\int_{-\infty}^{-0}F(x,y)e^{-i\eta y}dy+\int_{+0}^{+\infty}F(x,y)e^{-i\eta y}dy, \tag{22}$$

$$\bar{\bar{F}}(\xi,\eta)=\int_{-\infty}^{\infty}\bar{F}(x,\eta)e^{-i\xi x}dx.$$

We then treat Eqs. (10)—(12), (20) and (21) in the conventional manner and obtain the following descriptions of $\bar{\bar{\Psi}}(\xi,\eta)$, $\bar{\bar{X}}(\xi,\eta)$, and $\bar{\bar{p}}(\xi,\eta)$:

(i) for crossed fields,

$$\begin{pmatrix} \tilde{\tilde{A}} \\ \tilde{\tilde{\Psi}} \\ \tilde{\tilde{p}} \end{pmatrix} = \frac{R_m}{\Delta_\perp} \begin{pmatrix} i\xi\eta^2+(1-M^2)i\xi^3, & \eta^2-M^2\xi^2 \\ \frac{i\eta}{A^2}(\xi^2+\eta^2), & -i\eta\left\{\frac{1}{R_m}(\xi^2+\eta^2)+i\xi\right\} \\ -\frac{\xi^2}{A^2}(\xi^2+\eta^2), & \xi^2\left\{\frac{1}{R_m}(\xi^2+\eta^2)+i\xi\right\} \end{pmatrix} \begin{pmatrix} \frac{\tilde{h}}{R_m}+2\tilde{f} \\ \frac{\tilde{h}}{A^2}-2i\xi\tilde{f} \end{pmatrix} \quad (23)$$

where

$$\Delta_\perp=\eta^4(K+i\xi)+\eta^2\{i\xi^3(2-M^2)+K(1-A^2-M^2)\xi^2\}+(1-M^2)i\xi^5+K\xi^4(A^2M^2-A^2-M^2);$$

(ii) for aligned fields,

$$\begin{pmatrix} \tilde{\tilde{A}} \\ \tilde{\tilde{\Psi}} \\ \tilde{\tilde{p}} \end{pmatrix} = \frac{R_m}{\Delta_\parallel} \begin{pmatrix} \eta^2+(1-M^2)\xi^2, & -\xi\eta, & i\xi(1-M^2) \\ \frac{1-M^2}{A^2}(\xi^2+\eta^2), & i\eta\left\{\frac{\xi^2+\eta^2}{R_m}+i\xi\right\}, & (1-M^2)\left\{\frac{\xi^2+\eta^2}{R_m}+i\xi\right\} \\ -\frac{i\eta}{A^2}(\xi^2+\eta^2), & \frac{i\xi}{A^2}(\xi^2+\eta^2) -\xi^2\left\{\frac{\xi^2+\eta^2}{R_m}+i\xi\right\}, & -i\eta\left\{\frac{\xi^2+\eta^2}{R_m}+i\xi\right\} \end{pmatrix} \begin{pmatrix} \frac{i\eta}{R_m}\tilde{\tilde{m}} \\ 2\tilde{\tilde{f}} \\ -\frac{i\eta}{A^2}\tilde{\tilde{m}} \end{pmatrix} \quad (24)$$

where

$$\Delta_\parallel=\eta^4+\eta^2\{(2-M^2)\xi^2+iK(M^2+A^2-1)\xi\}+(1-M^2)\xi^4+iK(M^2-1)(1-A^2)\xi^3.$$

The inversion with respect to η of the foregoing transforms can be carried out by contour integration. The results are:

(i) for crossed fields,

$$\Psi(x,y)=\frac{1}{4\pi}\frac{y}{|y|}\sum_{l=1}^{2}(-1)^l \int_{-\infty}^{\infty}\frac{\tilde{f}c_{1l}+\tilde{h}c_2}{Z_2}e^{i(\xi X+\eta_l|Y|)}d\xi, \quad (25)\text{a}$$

$$X(x,y)=\frac{1}{4\pi}\sum_{l=1}^{2}(-1)^l \int_{-\infty}^{\infty}\frac{\tilde{f}e_{1l}+\tilde{h}e_{2l}}{Z_2}e^{i(\xi X+\eta_l|Y|)}d\xi, \quad (25)\text{b}$$

where

$$c_{1l}=-g_1(\xi^2+\eta^2_l)+KA^2\xi^2,$$

$$c_2=\frac{i}{z}K\xi,\ g_1=K+i\xi,$$

$$e_{1l}=-KA^2\frac{\zeta^3}{\eta_l}$$

$$e_{2l}=[\{(1-M^2)i\xi-KM^2\}\xi^2+g_1\eta_l^2]\frac{i}{2\eta_l},$$

$$g_2=\frac{\xi^2}{z}\{(2-M^2)i\xi+K(1-A^2-M^2)\},$$

$$g_3=\{(1-M^2)i\xi+K(A^2M^2-A^2-M^2)\}\xi^4,$$

$$\eta_l=\sqrt{\{-g_2+(-1)^l Z_2\}/g_1},$$

$$Z_2=\sqrt{g^2_2-g_1g_3};$$

(ii) for aligned fields,

$$\Psi(x,y)=\frac{1}{4\pi}\frac{y}{|y|}\sum_{l=1}^{2}(-1)^l \int_{-\infty}^{\infty}\frac{\tilde{f}b_{1l}+mb_2}{S_2}e^{i(\xi X+\eta_l|Y|)}d\xi, \quad (26)\text{a}$$

$$X(x,y)=\frac{1}{4\pi}\frac{y}{|y|}\sum_{l=1}^{2}(-1)^l \int_{-\infty}^{\infty}\frac{\tilde{f}a_1+ma_{2l}}{S_2}e^{i(\xi X+\eta_l|Y|)}d\xi, \quad (26)\text{b}$$

where

$$b_{1l}=-(KA^2i\xi+\xi^2+\eta_l^2),$$

$$b_2=\frac{K}{2}(1-M^2)i\xi,$$

$$a_1=-KA^2i\xi,$$

$$a_{2l}=-\frac{1}{2}\{(M^2-1)(iK\xi-\xi^2)+\eta_l^2\},$$

$$\eta_l=\sqrt{-\frac{1}{2}(2-M^2)\xi^2+\frac{K}{2}(1-M^2-A^2)i\xi+(-1)^l S_2},$$

$$S_2=\sqrt{\frac{1}{4}\{M^4\xi^4-K^2(M^2+A^2-1)^2\xi^2\}+\frac{KM^2}{2}(1+A^2-M^2)i\xi^3}.$$

Since the contour integration with respect to η picks up, for $y>0$, only the residues of poles in the upper half plane, and for $y<0$, only the residues of poles in the lower half plane, the square roots defining η_l must satisfy the condition

$I_m\sqrt{}>0$, for all ξ on the final inversion path. (27)

The implications of Eq. (25) and Eq. (26) are independent of the particular choice of the branches of the square roots Z_2 and S_2. For convenience we shall always choose the branch of any square root according to the condition (27).

When $M=1$ and the applied fields are aligned, Eq. (11) shows that the solution is of the form $\Psi(x, y)=\Psi_1(x)+\Psi_2(y)$. Hence, no solution exists for which the disturbance vanishes at infinity. Note that the integrands of Eqs. (25) and (26) are not singular at the zeros of Z_2 or S_2. Note also that when $R_m\to 0$ and/or $A\to\infty$, we recover the familiar results of conventional gasdynamics.

THE FLOW PAST A WAVY PLATE OR WALL

For many configurations the integrals in the general solutions (25) and (26) can be evaluated immediately. We shall study three such configurations. (1) The flow past a wavy plate with symmetrical thickness distribution. (2) The flow past a wavy plate with constant thickness. (3) The flow past a wavy wall occupying a half-space (See Figs. 5 and 6). The maximum non-dimensional thickness of the plate or the amplitude of the wavy wall surface is of order ϵ.

In the following, whenever l and/or n appear only on the right-hand side of an equation, it is implied that the expression be summed over $l=1$, 2 and/or over $-\infty<n<\infty$. No such summation is implied when an index appears on both sides of the equation.

Case(1). The boundary conditions to be used in Eq. (25) and Eq. (26) may be written as

$$f(x)=f_n e^{i\xi_n x}, \qquad \tilde{f}(\xi)=2\pi f_n\delta(\xi-\xi_n),$$

$$h(x)=h_n e^{i\xi_n x}, \qquad \tilde{h}(\xi)=2\pi h_n\delta(\xi-\xi_n),$$

$$m(x)=m_n e^{i\xi_n x}, \qquad \tilde{m}(\xi)2\pi_m m_n\delta(\xi-\xi_n),$$

where $\xi_n=2n\pi$, and $\delta(x)$ is Dirac δ-function.

For crossed fields, the solution (25) is

$$\begin{aligned}\Psi(x, y)&=\frac{y}{|y|}\Psi_{ln}e^{i(\xi_n x+\eta_{ln}|y|)},\\ X(x, y)&=X_{ln}e^{i(\xi_n x+\eta_{ln}|y|)},\\ p(x, y)&=p_{ln}e^{i(\xi_n x+\eta_{ln}|y|)},\\ j(x, y)&=j_{ln}e^{i(\xi_n x+\eta_{ln}|y|)},\end{aligned}\tag{29}$$

where

$$p_{ln}=i\xi_n^2\Psi_{ln}/\eta_{ln}, \qquad j_{ln}=(\xi^2_n+\eta_{ln}^2)A_{ln},$$

$$\Psi_{ln}=(-1)^l(f_n c_{1ln}+h_n c_{2n})/(2Z_{2n}),$$

$$X_{ln}=(-1)^l(f_n e_{1ln}+h_n e_{2ln})/(2Z_{2n}),$$

and where $c_{1ln}\equiv c_{1l}(\xi=\xi_n)$, $Z_{2n}\equiv Z_2(\xi=\xi_n)$, etc. The pressure drag D_p and the magnetic drag D_m per unit wave length of the plate, measured in units of $\rho_0U^2L\epsilon^2$, are

$$D_p=2i\xi_{-n}f_{-n}p_{ln}, \tag{30}$$

$$D_m=\frac{1}{A^2}i\xi_n A_{ln}h_{-n}. \tag{31}$$

For aligned fields, the solution is

$$\begin{aligned}\Psi(x, y)&=\frac{y}{|y|}\Psi_{ln}e^{i(\xi_n x+\eta_{ln}|y|)},\\ X(x, y)&=\frac{y}{|y|}X_{ln}e^{i(\xi_n x+\eta_{ln}|y|)},\\ p(x, y)&=p^l{}_n e^{i(\xi_n x+\eta_{ln}|y|)},\\ j(x, y)&=\frac{y}{|y|}j_{ln}e^{i(\xi_n x+\eta_{ln}|y|)},\end{aligned}\tag{32}$$

where

$$P_{ln}=i\eta_{ln}\Psi_{ln}/(M^2-1),$$

$$j_{ln}=(\xi^2_n+\eta^2_{ln})A_{ln},$$

$$\Psi_{ln}=(-1)^e(f_n b_{1ln}+m_n b_{2n})/(2S_{zn}),$$

$$X_{ln}=(-1)^e(f_n a_{1n}+m_n a_{2ln})/(2S_{zn}),$$

and where $b_{1ln}\equiv b_{1l}(\xi=\xi_n)$, $S_{zn}\equiv S_z(\xi=\xi_n)$, etc. The pressure drag is given by Eqs. (30) and (32), while the magnetic drag is

$$D_m=-\frac{1}{A^2}\xi_{-n}m_{-n}A_{ln}\eta_{ln}. \tag{33}$$

Case(2). The solutions obtained above can be used to find the flow past a wavy plate with constant thickness(of order ϵ), by simply multiplying by $\frac{y}{|y|}$ and redefining the magnetic singularities $h(x)$ and $m(x)$ in the body to satisfy the magnetic boundry conditions (20). The symmetrical solutions are now obtained in the form

(i) for crossed fields

$$\begin{cases} \Psi(x,-y)=\Psi(x,y), \\ \qquad p(x,-y)=-p(x,y), \\ A(x,-y)=-A(x,y), \\ \qquad h(x)=0; \end{cases} \tag{34 a}$$

(ii) for aligned fields

$$\begin{cases} \Psi(x,-y)=\Psi(x,y), \\ \qquad p(x,-y)=-p(x,y), \\ A(x,-y)=A(x,y), \\ \qquad m(x)=0. \end{cases} \tag{34 b}$$

The solution for Case (2) is obtained by determining the functions $h(x)$ or $m(x)$ in the solution of Case (1) in terms of $m(x)$ or $h(x)$ of Case (2) by Eq. (34) and the magnetic boundary conditions (20):

(i) for crossed fields,

h_n of Case (1) is to be replaced by

$$\frac{m_n Z_{2n}+f_n(e_{12n}-e_{11n})}{e_{21n}\ e_{22n}}; \tag{35 a}$$

(ii) for aligned fields,

m_n of Case (1) is to be replaced by

$$\frac{ih_n S_{2n}+f_n(\eta_{1n}-\eta_{2n})a_{1n}}{\eta_{2n}a_{22n}-\eta_{1n}a_{21n}}. \tag{35 b}$$

The flow fields are given by Eqs. (29), (32) and (35) for $y>0$, and by Eq. (34) for $y<0$. The pressure drags are given by Eq. (30), while the magnetic drags are given by Eq. (33) for crossed fields and by Eq. (31) for aligned fields, the substitutions (35) always being implied.

It should be noted that the flow fields for Cases (1) and (2) are independent of the magnetic permeability μ_w of the plate.

Case (3). The flow past an infinite wavy wall with magnetic permeability μ_w that occupies the half-space $y<0$ can be treated in a similar way. The magnetic field in the wall $(y<0)$ is current-free and may be written in the form

$$X\omega(x, y)=w_n e^{i\xi_n x+|\xi_n|y}, \quad y<0. \tag{36}$$

Let an artificial current sheet be provided in the wall at $y=-0$ such that

$$h_w(x)=h_{wn}e^{i\xi_n x}. \tag{37}$$

The magnetic boundary conditions at the inter-face $y=0$ can be written as

$$X(x,0)=X_w(x,0),$$

$$\frac{\partial X(x,0)}{\partial y}=\frac{\mu}{\mu\omega}\cdot\frac{\partial X_w(x,0)}{\partial y}-h_w(x). \tag{38}$$

Eq. (38) implies that X and X_w are nondimensionalized by the same quantity $B\epsilon$ and that h and hω are also nondimensionalized by the same quantity $\frac{B}{\mu}\epsilon$.

The flow fields, for $y>0$, of Case (1) satisfy the differential equations and hydrodynamic boundary conditions of Case (3), for arbitrary function $h(x)$ or $m(x)$ of Case (1). This function and the magnetic field in the wall are determined by Eq. (38). The results are:

(i) for crossed fields,

$$w_n=[(e_{22n}-e_{21n})\{h_n \text{ of Case}(1)\} + (e_{12n}-e_{11n})f_n]/(2Z_{2n}), \tag{39}$$

h_n of Case (1) is to be replaced by

$$\frac{2Z_{2n}h_{wn}-|\xi_n|(e_{12n}-e_{11n})f_n\frac{\mu}{\mu_w}}{Z_{2n}+|\xi_n|(e_{22n}-e_{21n})\frac{\mu}{\mu_w}}; \tag{40}$$

(ii) for aligned fields,

$$w_n=-\frac{1}{2}\{m_n \text{ of Case (1)}\}, \tag{41}$$

m_n of Case (1) is to be replaced by

$$\frac{ia_{1n}(\eta_{1n}-\eta_{2n})f_n-2S_{2n}h_{wn}}{i(a_{22n}\eta_{22n}-a_{21n}\eta_{21n})+|\xi_n|S_{2n}\frac{\mu}{\mu_w}}. \tag{42}$$

We shall normalize the drag per wave length of the wavy wall by $\frac{1}{2}\rho_0 U^2 L\epsilon^2$, so that Eq. (30) is still valid as the pressure drag with the substitutions (40) or (42). The magnetic drag on the wall due to the imbedded current distribution $h_w(x)$ is computed as:

$$D_m = \frac{2}{A^2} i\xi_n w_n h_{w-n} \qquad (43)$$

for both crossed and aligned and aligned fields.

An immediate consequence of Eqs. (35), (40) and (42) is that, when there are no magnetic singularities in the bodies, the solutions for $y>0$ for the various configurations are related by:

(i) crossed fields

$$\begin{cases} \text{Case (1) (for arbitrary } \mu_w) \\ \qquad = \text{Case (3) (for } \mu_w=\infty), \\ \text{Case (2) (for arbitrary } \mu_w) \qquad (44)\text{a} \\ \qquad = \text{Case (3) (for } \mu_w=0); \end{cases}$$

(ii) aligned fields

$$\begin{cases} \text{Case (1) (for arbitrary } \mu_w) \\ \qquad = \text{Case (3) (for } \mu_w=0), \\ \text{Case (2) (for arbitary } \mu_w) \qquad (44)\text{b} \\ \qquad = \text{Case (3) (for } \mu_w=\infty). \end{cases}$$

The incompressible flow theory of Broer and van Wijngaarden (1963) for Case (1), (2) and (3) is obtained as the case $M=0$ of the present theory. The drag curves of Resler and McCune (1959) for Case (3) can be reproduced by putting $M=0$, $\mu=\mu_w$, and $h_w(x)=0$ in Eq. (30). The present theory reduces to that of Sears and Resler (1959) when $M=0$ and $R_m=\infty$, and to that of Ackeret when $R_m=0$ or $A=\infty$.

MHD waves and magnetic boundary layers. Summations over l and n always give real numbers in all expressions above. To each term with subscripts l and n, there is a complex conjugate with subscripts $l+1$ (or $l-1$) and $-n$ in the summation. Therefore the following are equivalent summations:

$$\sum_{l=1}^{2} \sum_{n=-\infty}^{\infty} = 2R_e \sum_{n=-\infty}^{\infty} \quad (l=1 \text{ or } 2)$$

$$= 2R_e \sum_{l=1}^{2} \sum_{n=1}^{4} \qquad (45)$$

For each Fourier compoent of the boundary conditions, i. e., for fixed $|n|$, there are two terms in the solution. These two terms are generalizations of the fast and slow waves of the infinite R_mMHD theory, and represent two damped waves in the solutions (29) and (32). The damping distances δ_{ln} from the wavy surface of the body over which the flow perturbations decrease to $\frac{1}{e}$ of that at the body surface are

$$\delta_{ln} = [I_m(\eta_{ln})]^1. \qquad (46)$$

The tangents of the wave angles (Mach angle when $R_m \to 0$) are

$$\left(\frac{dy}{dx}\right)_{ln} = -\xi_n [R_e(\eta_{ln})]^{-1}. \qquad (47)$$

δ_{ln} is always positive due to the condition (27). The damped wave is facing downstream (upstream) if $\left(\frac{dy}{dx}\right)_{ln}>0$ (<0). Figures 1 and 2 are wave diagrams obtained from Eqs. (46) and (47) for sinusoidal boundary conditions [See (53)].

Each of the wave diagrams (Figs. 1 and 2) consists of two curves corresponding to the two terms meationed above. For given A, M, and R_m, vertical coordinates of these curves are fixed by the damping distances δ, while the wave angles are

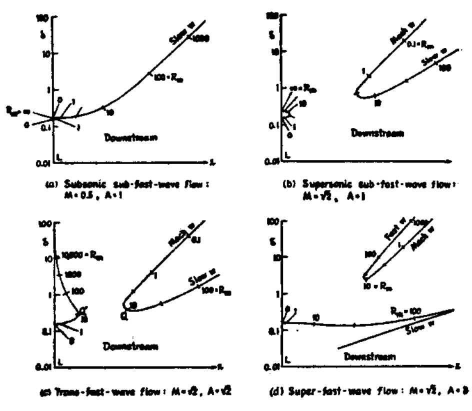

Fig. 1. Wave diagvams for the case, where U is perpendicular to B

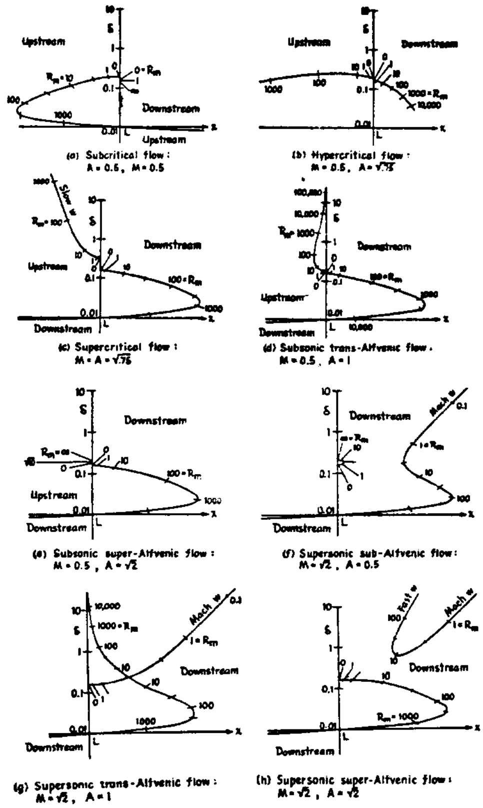

Fig. 2. (a)-(h) wave diagrams for the case, where U is aligned with B.

used as the polar coordinates of the wave diagram. Numbers in the wave diagrams are the magnetic Reynolds numbes R_m, some of which differing by a factor $\sqrt{10}$ from the adjacent are not shown. In Fig. 1(c), for example, the wave diagram shows that, for crossed fields with $A=M=\sqrt{2}$ and $R_m=10$, the two damped waves, denoted by Q and Q′, have damping distances $\delta=0.386$, $\delta'=0.291$, and wave angles $\angle QLx=39.7°$, $\angle Q'Lx=66.7°$, respectively. One of the two waves in general becomes a Mach wave in the limit $R_m\to 0$; the amplitudes Ψ_{ln}, $X_{l\mu}$, etc. of the other wave vanish as $R_m\to 0$. It is seen in the wave diagrams that the slow and fast waves of the infinite R_m theory appear as $R_m\to\infty$.

For crossed fields, Fig. 1 shows that the waves always face downstream. For aligned fields, however, Fig. 2 shows that, for subsonic flow ($M<1$), one of the two waves faces upstream and the other faces downstream; for supersonic flow ($M>1$), both waves face downstream. Another major distinction between Fig. 1 and Fig. 2 is the appearance for aligned fields (Fig. 2) of a "tail" that passes the point L and implies that, for one of the two waves, the damping distance δ approaches zero and the wave angle approaches 180° for subcritical flow ($M^2+A^2<1$) and approaches 0° for $M^2+A^2>1$ in the limit $R_m\to\infty$. This is the MHD boundary layer (or MHD wake) generalized to compressible flow. The wake is facing upstream for subcritical flow and downstream otherwise. In fact, one can expand Eqs. (46) and (47) at large R_m to obtain expressions for the MHD boundary layer

for aligned fields and $R_m\to\infty$,

$$\delta=\left(\frac{2A}{R_m|M^2+A^2-1|}\right)^{1/2}\cdot\frac{1}{|\xi_n|} \quad (48)$$

$$\left(\frac{dy}{dx}\right)\simeq\left(\frac{2A}{R_m|M^2+A^2-1|}\right)^{1/2}\cdot\frac{M^2+A^2-1}{|M_2+A^2-1|}, \quad (49)$$

and the infinite R_m theory. For incompressible flow, Eqs. (46) and (47) for aligned fields simplify to the irrotational flow

$$\delta=\frac{1}{|\xi_n|},\quad \left(\frac{dy}{dx}\right)=\infty, \quad (50)$$

and an MHD boundary layer

$$\delta=\left[\frac{2}{\{\xi_n{}^4+K^2\xi_n{}^2(A^2-1)^2\}^{1/2}+\xi_n{}^2}\right]^{1/2}$$

$$\left(\frac{dy}{dx}\right)=\left[\frac{2}{\{\xi^4+K^2\xi_n{}^2(A^2-1)^2\}^{1/2}-\xi_n{}^2}\right]^{1/2}|\xi_n|\frac{A-1}{|A-1|}, \quad (51)$$

that faces upstream for sub-Alfvenic flow ($A<1$) and downstream for super-Alfvenic flow ($A>1$). When $M=0$, Eq. (48) reduces to the formula of Sears and Resler (1959) for the thickness of MHD boundary layer in incompressible flow. The idea of an upstream facing wake has so far been discussed only for incompressible flow (Hasimoto 1959, Greenspan and Carrier 1959, and Lary 1962). Existence of such a wake in compressible flow has been predicted by Resler and McCune (1959), and is confirmed in Eqs. (48), (49), and in the wave diagram Fig. 2(a).

As $R_m \to \infty$, the perturbations in the thin MHD boundary layer become so large that one must either require ϵ to be very small or use a nonlinear theory. This limitation of the present theory for aligned is most severe for the configuration of Case (1). Figure 2 shows, however, that the MHD boundary layer phenomenon is quite diffuse when R_m is around 10.

Figure 2(c) shows another drastic departure from ordinary gasdynamics by having an upstream facing wave for large R_m for supercritical flow ($M^2+A^2>1$, $M<1$, $A<1$). This phenomenon has been discussed in the infinite R_m theory of Resler and McCune (1959), Kogan (1959), and Sears (1961). Exact solutions of compressible conducting flow past a thin symmetrical airfoil for $R_m=\infty$ will be presented later in the present paper.

MHD drag. Expressions for the drag on the body are presented in Eqs. (30), (31), (33), and (43). When there are no magnetic singularities in the body, the drag is given by the pressure drag (30). In Figs. 3 and 4, the drag coefficient C_D

$$C_D \equiv \frac{D_p}{(2\pi)^2} \qquad (52)$$

for the sinusoidal wavy plate

$$f(x) = \epsilon \cos 2\pi x \qquad (53)$$

of Case (2) without magnetic singularities in the plate is plotted versus R_m for various of M and A. For crossed fields, Fig. 3 shows that the drag is greater than that of conventional flow for subsonic flow and is less than that of conventional flow for supersonic flow. For supersonic flow, C_D decreases with increased magnetic field for crossed fields but increases for aligned fields (Figs. 3 and 4). This dependence of C_D on the magnetic field can be seen more clearly in Figs. 5 and 6, where C_D is plotted versus A for various of M for all configurations. It is seen that, except for supersonic flow with crossed fields, the drag increases indefinitely with increasing magnetic field.

INFINITE R_m RESULTS FOR THIN SYMMETRICAL AIRFOIL

Compressible MHD flows past a flat airfoil for $R_m=\infty$ have been studied by Resler and McCune (1959) and Sears (1961). For $R_m=\infty$, the integrals in the general solutions (25) and (26) for a thin symmetrical airfoil can be evaluated. We shall write down the results of integrations in the following.

Crossed fields. Let us first define $k(x)$ by

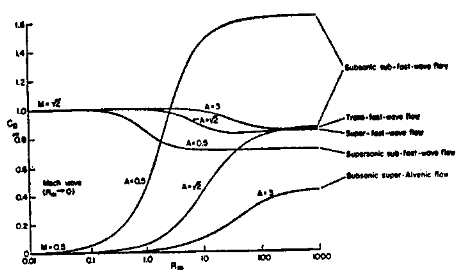

Fig. 3. Drag coefficient Co of wavy plate with zero thickness plotted vs. R_m for various values of Mach number M and Alfven number A for the case, where U is perpendicular to B.

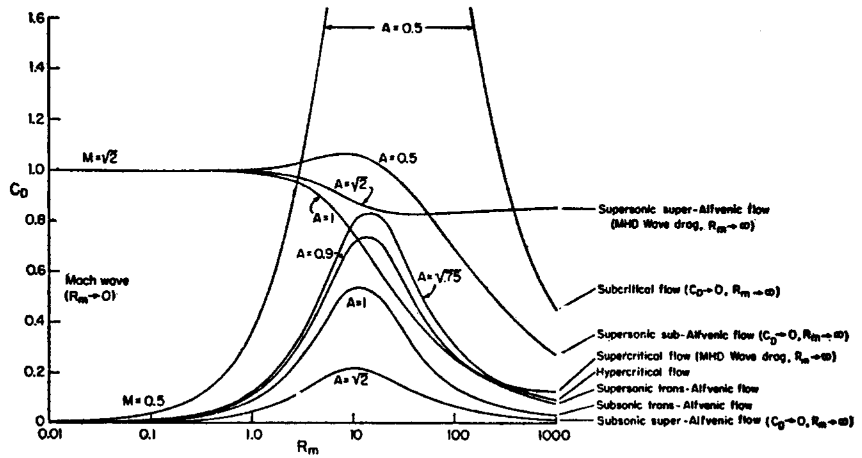

Fig. 4. Drag coefficient C_D of wavy plate with zero thickness vs. R_m for various values of M and A for the case, where U is aligned with B.

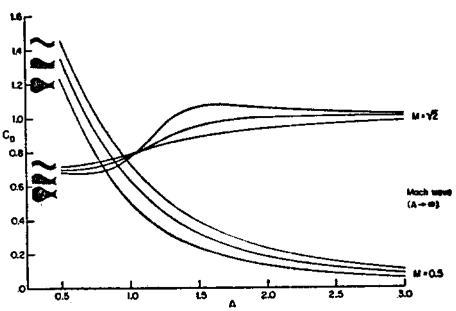

Fig. 5. Drag coefficient C_D of wavy wall with $\mu=\mu_w$ ( ), wavy plate with zero thickness ( ) and symmetrical wavy plate ( ) plotted vs. Alfven number A fo various values of Mach number M for the case, where $R_m=IC$ and U is perpendicular to B.

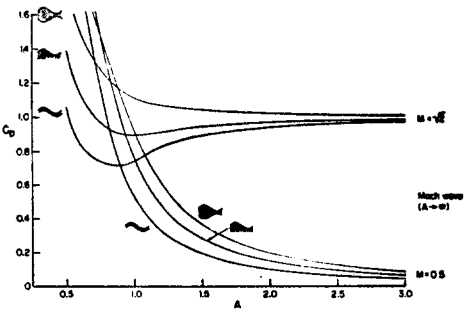

Fig. 6. Drag coefficient C_D of wavy wall with $\mu=\mu_w$ ( ), wavy plate with zero thickness ( ) and symmetrical wavy plate ( ) plotted vs. Alfven number A for various values of Mach number M for the case, where $R_m=10$ and U is aligned with B.

$$\frac{dk(x)}{dx}=h(x),\ k(a)=0, \qquad (54)$$

so that the last condition of (20) is automatically satisfied. We shall use the subscript sub when the velocity of the free stream is less than that of the fast wave ($M^{-2}+A^{-2}>1$), and super when the velocity of the free stream is greater than that of the fast wave ($M^{-2}+A^{-2}<1$). The results of integrations are

$$\Psi_{sub}(x,\ y)=\frac{y}{|y|}\Big\{R_2(x-C_2|y|)$$

$$-\frac{C_0}{\pi}|y|\int_a^b\frac{R_1(t)\,dt}{(x-t)^2+C_0^2y^2}\Big\},$$

$$\Psi_{super}(x,\ y)=\frac{y}{|y|}\{R_2(x-C_2|y|)-R_1(x-C_1|y|)\}, \qquad (55)$$

$$P_{sub}=-\Big\{\frac{1}{C_2}R'_2(x-C_2|y|)+\frac{1}{\pi C_0}\int_a^b\frac{R'_1(t)\,(x-t)\,dt}{(x-t)^2+C_0^2y^2}\Big\},$$

$$P_{super}=-\Big\{\frac{1}{C_2}R'_2(x-C_2|y|)-\frac{1}{C_1}R'_1(x-C_1|y|)\Big\}, \qquad (56)$$

$$X_{sub}=-\left(C_2-\frac{M^2}{C_2}\right)R_2\ (x-C_2|y|)$$
$$+\frac{1}{\pi}\left(C_2+\frac{M^2}{C_0}\right)\int_a^b\frac{R_1(t)\,(x-t)\,dt}{(x-t)^2+C^2{}_0y^2},$$
$$X_{super}=-\left(C_2-\frac{M^2}{C_2}\right)R_2(x-C_2|y|)$$
$$-\left(\frac{M^2}{C_1}-C_1\right)R_1(x-C_1|y|), \quad (57)$$
$$j_{sub}=A^2\left\{\left(\frac{1-M^2}{C_2}+C_2\right)R''_2(x-C_2|y|)\right.$$
$$\left.-\frac{1}{\pi}\left(\frac{M^2-1}{C_0}+C_0\right)\int_a^b\frac{R''_1(t)\,(x-t)\,dt}{(x-t)^2+C^2{}_0y^2}\right\},$$
$$j_{super}=A^2\left\{\left(\frac{1-M^2}{C_2}+C_2\right)R''_2(x-C_2|y|)\right.$$
$$\left.+\left(\frac{M^2-1}{C_1}+C_1\right)R''_1(x-C_1|y|)\right\} \quad (58)$$

where

$$R_1(x)\equiv\frac{C_4f(x)-k(x)}{2C^2},$$
$$R_2(x)\equiv\frac{C_3f(x)-k(x)}{2C^2}=R_1(x)-f(x),$$
$$C_3\equiv A^2-M^2-1-C^2<0,$$
$$C_4\equiv C_3+2C^2>0,\ C_0\equiv\sqrt{-C_1^2} \quad (59)$$

The first terms (C_2 terms) of Eqs. (55)-(58) are recognized as the slow wave and the second terms (C_0 or C_1 terms) the fast wave. An immediate observation in Eq. (58) is that the elliptic part of the current vanishes for incompressible flow ($M=0$).

The pressure drag on the airfoil normalized by $e_0\ U_2\ L\epsilon^2$ are computed from Eq. (56) as

$$D_{p\ sub}=-\frac{1}{C_2}\left[\frac{1}{C_2}\int_a^b\{C_3(f'(x))^2-k'(x)\right.$$
$$\left.f'(x)\}\,dx-\frac{1}{\pi C_0}\int_a^b\int_a^b\frac{f'(x)\,k'(x)}{x-t}\,dx\,dt\right]$$
$$D_{p\ super}=-\frac{1}{C_2}\left[\frac{1}{C_2}\int_a^b\{C_3(f'(x))^2\right.$$
$$-k'(x)f'(x)\}\,dx-\frac{1}{C_1}$$
$$\left.\int_a^b\{C_4(f'(x))^2-k'(x)f'(x)\}\,dx\right]. \quad (60)$$

The magnetic drag on the airfoil is computed from Eq. (57) as

$$D_{m\ sup}=-\frac{1}{2C^2A^2}\left[\left(C_2-\frac{M^2}{C_0}\right)\right.$$
$$\int_a^b\{C_3f'(x)\,k'(x)-(k'(x))^2\}\,dx$$
$$+\frac{C_4}{\pi}\left(C_0+\frac{M_2}{C_0}\right)$$
$$\left.\int_a^b\int_a^b\frac{f'(x)\,k'(t)}{x-t}\,dx\,dt\right], \quad (61)$$
$$D_{m\ super}=-\frac{1}{2C^2A^2}\left[\left(C_2-\frac{M^2}{C_2}\right)\right.$$
$$\int_a^b\{C_3f'(x)\,k'(x)-(k'(x))^2\}\,dx$$
$$+\left(\frac{M^2}{C_1}-C_1\right)\int_a^b\{C_4f'(x)\,k'(x)$$
$$\left.-(k'(x))^2\}\,dx\right].$$

Equations (60) and (61) provide a variational problem for minimizing the total drag $D=D_p+D_m$ by choosing the optimal $k(x)$ for a given $f(x)$. Here, we simply demonstrate that the present theory can have negative drag ($D<0$) for sub-fast-wave flow. Suppose the current in the body is related to the geometry by $h(x)=\alpha f'(x)$, i.e., $k(x)=\alpha f(x)$, where α is a real number. Then the elliptic terms (fast wave terms) of Eq. (60) and Eq (61) vanish.

Now, $D_{p\ sub}=0$ and $D_{m\ sub}=0$ when $\alpha=\alpha_1=C_3$; and $D_{p\ sub}+D_{m\ sub}=D_{sub}=0$ when $\alpha=\alpha_2=-\dfrac{2A^2}{C^2{}_2-M^2}$. Thus, D_{sub} is a quadratic function of α which vanishes at $\alpha=\alpha_1$, and α_2, while $D_{sub}<0$ for α between α_1 and α_2. Notice that, always, $\alpha_1<0$ and $\alpha_2<0$. Also, it is easy to see that $\alpha_1\rightarrow-2$, $\alpha_2\rightarrow-2$ when $M\rightarrow0$, and the flow fields become irrotational and current-free. Therefore zero drag is minimum drag for the group $h(x)=\alpha f'(x)$ in incompressible flow with crossed fields.

When $M=0$ and $k(x)=0$, Eq. (60) reduces to the Alfven wave drag of Sears and Resler (1959). For $A=\infty$, Eq. (60) reduces to the Mach wave drag of Ackeret.

Aligned fields. Let us define the subscripts 1 through 4 to denote the various regions of the Taniuti-Resler diagram by

(1) subcritical flow

$M^2+A^2<1$;

(2) supercritical flow

$M^2+A^2>1$; $M<1$, $A<1$;

(3) subsonic, super-Alfvenic

$M<1$, $A>1$;

or supersonic, sub-Alfvenic flow

$M>1$, $A<1$;

(4) supersonic, super-Alfvenic flow

$M>1$, $A>1$.

Then Eq. (24), after inversion, becomes

$$X_k(x, y)=\Psi_k(x, y), \quad k=1, 2, 3, 4, \quad (62)$$

$$\left.\begin{aligned} \Psi_1(x, y) &= \Psi_3(x,y) \\ &= -\frac{F}{\pi}y\int_a^b \frac{f(t)\,dt}{(x-t)^2+F^2y^2}, \\ \Psi_2(x, y) &= -\frac{y}{|y|}f(x+F|y|), \\ \Psi_4(x, y) &= -\frac{y}{|y|}f(x-F|y|), \end{aligned}\right\} \quad (63)$$

$$\left.\begin{aligned} p_1(x, y) &= p_3(x, y) \\ &= \frac{F}{(M^2-1)\pi}\int_a^b \frac{f'(t)\,(x-t)\,dt}{(x-t)^2+F^2y^2}, \\ p_2(x, y) &= \frac{F}{1-M^2}f'(x+F|y|), \\ p_4(x, y) &= \frac{F}{M^2-1}f'(x-F|y|), \end{aligned}\right\} \quad (64)$$

$$\left.\begin{aligned} j_1(x, y) &= j_3(x, y) = \frac{A^2M^2y}{\pi(M^2+A^2-1)} \\ &\quad \times\int_a^b \frac{f''(t)\,dt}{(x-t)^2+F^2y^2}, \\ j_2(x, y) &= \frac{A^2M^2}{M^2+A^2-1} \\ &\quad \times\frac{y}{|y|}f''(x+F|y|), \\ j_4(x, y) &= \frac{A^2M^2}{M^2A^2-1} \\ &\quad \times\frac{y}{|y|}f''(x-F|y|), \end{aligned}\right\} \quad (65)$$

where

$$F=\left|\frac{(1-M^2)(1-A^2)}{1-M^2-A^2}\right|^{1/2}$$

and

$$m(x)=2f(x). \quad (66)$$

This condition (66) follows from the hydrodynamic boundary condition $\Psi(x,\pm 0)=\mp f(x)$ after η inversion of Eq. (24) with $R_m=\infty$. Equation (62) shows that the magnetic flux lines are aligned with the streamlines. When the magnetic singularities in the airfoil do not satisfy Eq. (66), a linearized solution must have an equivalent current sheet on the surface of the body such that the discontinuity in $b_y(x, y)$ across the airfoil is $2f'(x)$. In view of the wave diagram in Fig. 2, this current sheet corresponds to the MHD boundary layer of finite R_m theory.

The pressure drag on the airfoil is computed from Eq. (64) as

$$D_{p1}=D_{p3}=0, \quad D_{p2}=D_{p4} = \frac{2F}{|1-M^2|}\int_a^b \{f'(x)\}^2dx. \quad (67)$$

The magnetic drag due to Eq. (66) is computed from Eqs. (62) and (63) as

$$D_{m1}=D_{m3}=0, \quad D_{m2}=-D_{m4} = -\frac{2F}{A^2}\int_a^b \{f'(x)\}^2dx. \quad (68)$$

The magnetic drag is negative for supercritical flow, and the total drag is nonnegative in any case.

Equation (64) shows that the compression region of subsonic elliptic flow is identical with the expansion region of supersonic elliptic flow, while the streamlines given by Eq. (63) are the same. The velocity-pressure variation along a stream tube for elliptic flow is similar to that of ordinary gasdynamics as has been pointed out by Sears (1960).

The flow past a wavy plate of Case (1) for $R_m=\infty$ can readily be obtained from Eqs. (55)~(58) and (62)~(66) by replacing the limits of integration (a, b) by $(-\infty, \infty)$ and carrying out the integration.

SIMILARITY SOLUTIONS OF INCOMPRESSIBLE FLOW

The Fourier transforms of the solutions for a thin symmetrical airfoil (23) and (24) can be used directly to determine the magnetic singularities $h(x)$ or $m(x)$ for desired flow fields. When these flow fields are invariant (except for constant factors) for a certain group of the set of parameters A, M, and R_m, we shall use the term similarity solution. We have already encountered a similarity solution relating supersonic sub-Alfvenic flow and subsonic elliptic flows in the previous section. Two similarity solutions of incompressible flow will be considered in the following.

Classical ideal flow. The classical ideal flow, i.e., inviscid incompressible nonconducting fluid flow is characterized by the Laplace equation. This corresponds to the case $M=0$, $R_m=0$ in Eqs. (23) and (24). It is straightforward to show that this can be an MHD solution for arbitrary R_m if

$$\left.\begin{aligned} &h(x)=-2f'(x) \\ &\qquad\text{for crossed fields;} \\ \text{and}\quad &m(x)=2f(x) \\ &\qquad\text{for aligned fields,} \\ \text{with}\quad &M=0,\ \text{arbitrary } A \text{ and } R_m. \end{aligned}\right\} \quad (69)$$

The stream function is given by

$$\Psi(x,y)=-\frac{y}{\pi}\int_a^b\frac{f(t)\ dt,}{(x-t)^2+y^2}, \quad (70)$$

and there is no current in the flow. It is easy to see that the magnetic singularities provided in the body by Eq. (69) are such that the electromotive force $E+V\times B$ in the flow vanishes identically.

The drag on the airfoil is zero; the Alfen wave drag is balanced out by the effect of the magnetic singularities (69) introduced in the airfoil.

An upstream wake. It is convenient to redefine the magnetic Reynolds number $R_m$as a real number, positive when the free stream velocity is in the positive x-direction and negative when the free stream velocity is in the negative x-direction. Then, the direction of the free stream velocity appears in the general solutions (23) and (24) only through R_m (or K). Equation (17), which may be called an MHD Oseen equation, shows that incompressible flow with aligned fields is governed by the same differential equation for the group $K(A^2-1)=$const. In particular, sub-Alfvenic flow ($K>0$, $A<1$) and super-Alfvenic reversed flow ($K<0$, $A>1$) are governed by the same differential equation (17). But, Eq. (24) is not invariant under the group $K(A^2-1)=$const. due to the "manner" in which the boundary conditions enter. However, if the body is magnetized according to $m(x)=2A^2f(x)$, a similarity solution is obtained as

$$\left.\begin{aligned} &A_1^2A(x,y;R_m,A) \\ &\quad=A^2A_1(x,y;R_{m1},A_1), \\ &\Psi(x,y;R_m,A) \\ &\quad=\Psi_1(x,y;R_{m1},A_1), \\ &p(x,y;R_m,A) \\ &\quad=p_1(x,y;R_{m1},A_1), \\ &K(A^2-1) \\ &\quad=K_1(A_1^2-1),\ M=M_1=0, \\ &m(x)=2A^2f(x), \\ &\quad m_1(x)=2A_1^2f(x). \end{aligned}\right\} \quad (71)$$

We have seen in Eq. (49) and Fig. 2 that, for an inviscid incompressible MHD flow with aligned fields, the upstream disturbance is bigger (smaller) than the downstream disturbance for sub(super)-Alfvenic flow. Then Eq. (71) gives a similarity law that relates the upstream wake of sub-Alfvenic reversed flow ($A<1, K<0$) to the downstream wake of super-Alfvenic flow ($A>1, K>0$). A similar theorem for viscous flow with $R_m=\infty$ has been found by Hasimoto (1959). To clarify the meaning of Eq. (71), consider a sub-Alfvenic flow and a super Alfvenic reversed flow related by Eq. (71) where the free streams have the same density ρ^0 and magnitude of velocity $|U|$. Then, the pressure fields (With physical dimensions) of the two systems of flow are identical, the velocity fields (with dimension) have the same streamlines and magnitudes of velocity but have oppsite flow direction everywhere, while the magnetic fields differ by a constant factor such that the physical Lorenz force of the two systems of flow are given by the same function of the coordinates x and y.

FLOW PAST WAVY WALL WITH FINITE THICKNESS

The solutions obtained above for flow past many plate may be used to construct flow past wavy wall with finite thickness where free stream conditions above and below the wall may be different. The configuration of the problem is shown in Fig. 7. The y coordinates of the upper and lower surfaces are $y_u(x)=t+f(x)$ and $y_l(x)=-t'-f'(x)$, respectively, $f(x)$, and the current sheet $i(x)$ imbedded in the wall at $y=0$ and normalized by B/μ_w can be written in terms of their Fourier expansion:

$$f(x)=f_n\exp(i\xi_n x)$$
$$i(x)=i_n\exp(i\xi_n x) \qquad (72)$$

For the crossed fields case the applied B is constant, and for the aligned fields case the applied field B/μ is constant.

The differential equations to be satified by the flow flieds are the same as in the case of a wavy plate with infinitesimal thickness.

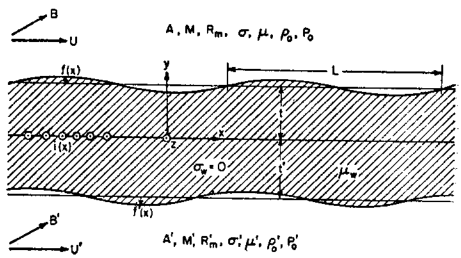

Fig. 7. Configuration of the proflem. B and B'lie either in the x or y direction.

Therefore, the solutions given by (28), (29), and (32) are still valid provided that the functions $h(x)$ and $m(x)$ are reinterpreted to satisfy the magneti boundary conditions at the interfaces $y=t$, o, and t'.

The magnetic field in the wall is harmonic and hence may be written as:

$$X_w(x,y)=(d_{1n}\sinh\xi_n y+d_{2n}\cosh\xi_n y)\times\exp(i\xi_n x) \quad \text{for} \quad 0<y<t. \qquad (73)$$

$$X'_w(x,y)=(d'_{1n}\sinh\xi_n y+d'_{2n}\cosh\xi_n y)\exp(i\xi_n x) \quad \text{for} \quad -t'<y<0.$$

For the magnetic field in the flow, we write, from (29) and (32)

$$X(x,y)=(F_{ln}f_l+K_{ln}k_n)\exp[i\{\xi_n x+\eta_{ln}(y-t)\}],\ y>t;$$
$$X'(x,y)=sg\ (F'_{ln}f'_n+K_{ln}k'_n)\exp[i\{\xi_n x-\eta'_{ln}(y+t')\}],\ y<-t' \qquad (74)$$

where, for crossed fields,

$$F_{ln}=(-1)^l\frac{e_{1ln}}{2Z_{2n}},\quad K_{ln}=(-1)^l\frac{e_{2ln}}{2Z_{2n}}$$
$$k_n=h_n.\quad sg=1;$$

and, for aligned fields.

$$F_{ln}=(-1)^l\frac{a_{1n}}{2S_{2n}},\quad K_{ln}=(-1)^l\frac{a_{2ln}}{2S_{2n}},$$

$k_n = m_n, \quad sg = -1.$

The magnetic boundary conditions at the inter-faces are

$$X_w = X'_w,$$
$$\frac{\partial X'_w}{\partial y} = \frac{\partial V_w}{\partial y} + i(x) \text{ at } y=0;$$
$$X_w = X,$$
$$\frac{1}{\mu_w}\frac{\partial X_w}{\partial y} = \frac{1}{\mu}\frac{\partial X}{\partial y} \text{ at } y=t \quad (75)$$
$$X'_w = X',$$
$$\frac{1}{\mu_w}\frac{\partial X'_w}{\partial y} = \frac{1}{\mu'}\frac{\partial X'}{\partial y} \text{ at } y=-t'$$

Solving (73)-(75) for $d_{1n}, d_{2n}, d'_{1n}, d'_{2n}, k_n, k'_n$, gives

$$d'_{2n} = d_{2n}, \quad d'_{1n} = d_{1n} + \frac{i_n}{\xi_n} \quad (76)$$

$$\begin{pmatrix} d_{1n} \\ d_{2n} \\ k_n \\ k'_n \end{pmatrix} = \Delta_n^{-1} \begin{pmatrix} f_n(F_{ln}) \\ f_n \frac{i\mu_w}{\xi_n \mu}(\eta_{ln}F_{ln}) \\ f'_n(F'_{ln})sg + \frac{i_n}{\xi_n}\sinh \xi_n t' \\ f'_n \frac{i\mu_w}{\xi_n \mu'}(\eta'_{ln}F'_{ln})sg + \frac{i_n}{\xi_n}\cosh \xi_n t' \end{pmatrix} \quad (77)$$

$$\Delta_n = \begin{pmatrix} \sinh\xi_n t & \cosh\xi_n t & -(K_{ln}) & 0 \\ \cosh\xi_n t & \sinh\xi_n t & -\frac{i\mu_w}{\xi_n\mu}(\eta_{ln}K_{ln}) & 0 \\ -\sinh\xi_n t & \cosh \xi_n t' & 0 & -(K'_{ln})sg \\ -\cosh \xi_n t & \sinh \xi_n t' & 0 & -\frac{i\mu_w}{\xi_n\eta'}(\eta'_{ln}K'_{ln})sg \end{pmatrix} \quad (78)$$

where

$$(F_{ln}) = F_{1n} + F_{2n}, \quad (\eta_{ln}K_{ln}) = \eta_{1n}K_{1n} + \eta_{2n}K_{2n},$$

etc.

(73), (74), (76), (77) completely determine the magnetic field. The stream function $\Psi(x, y)$ and the pressure $p(x, y)$ are, in view of (29) and (32), given by,

$$\Psi(x, y) = \Psi_{ln}\exp[i\{\xi_n x + \eta_{ln}(y-t)\}], \ y>t,$$
$$p(x, y) = p_{ln}\exp[i\{\xi_n x + \eta_{ln}(y-t)\}], \ y>t;$$
$$\Psi(x, y) = -\Psi'_{ln}\exp[i\{\xi_n x - \eta'_{ln}(y+t')\}], \quad y<-t',$$
$$p(x, y) = p'_{ln}\exp[i\{\xi_n x - \eta'_{ln}(y+t')\}], \quad y<-t',$$

the substitution h_n (or M_n) $= k_n$ being implied always.

The solutions obtained by Broer & van Wijngaarden are contained in the present theory as the aligned fields cases:

$M=0, \ \mu=\mu'=\mu_w, \ A=A', \ R_m=R'_m,$

(1) $f(x)=f'(x), \ i(x)t=0, \ t=t'\to 0;$

(2) $f(x), = -f'(x), \ i(x)=0, \ t=t'\to 0;$

(3) $i(x)=0, \ t\to 00, \ t'-0;$ *and*

(4) $f(x)=f'(x), \ i(x)=0.$

The 3 special solutions obtained in (29)-(44) correspond to the cases:

$A=A', \ M=M', \ R_m=R'_m t,$

(1) $f(x)=f'(x), \ t=t'\to 0;$

(2) $f(x)=-f'(x), \ t=t'\to 0;$ and

(3) $t\to 00, \ t'=0.$

The structure of the damped waves in the flow fields are characterized by η_{ln} which has been extensively discussed in (45)-(51).

For aligned fields, the present theory breaks down as $R_m\to 00$ unless further conditions are imposed on $f(x)$, $i(x)$ and t, t' (see Refs. [1]-[14]

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REFERENCES

[1] Broer, L. J. F. and van Wijngaarden, L. *Magnetohydrodynamic Flow along Wavy Walls, Appl. Sci. Res.*, **B9**, 451 (1963).

〔2〕 H. P. Greenspan, and G. F. Carrier, *The Magnetohydrodynamic Flow Past a Flat Plate, J. Fluid Mech.* **6**, 77 (1959).

〔3〕 Ho. Hasimoto, *Viscous Flow of a Perfectly Conducting Fluid with a Frozed Magnetic Field, Phys. Fluids*, **2**, 338 (1959).

〔4〕 M. N. Kogan, *Magnetodynamics of Plane and Axisymmetric Flows of a Gas with Infinite Conductivity, Prikl. Mat. i. Mekh.* **23**, 70 (1959).

〔5〕 E. C. Lary, *A Theory of Thin Airfoils and Slender Bodies in Fluids of Finite Electrical Conductivity with Aligned Fields, J. Fluid Mech.* **12**, 209 (1962).

〔6〕 J. E. McCune, *On the Motion of Thin Airfoils in Fluids of Finite Electrical Conductivity, J. Fluid Mech.* **7**, 449 (1960).

〔7〕 J. E. McCune, and E. L. Resler Jr., *Compressibility Effects in Magnetoaerodynamic Flows past Thin Bodies, J. Aero/Space Sci.* **27**, 493 (1960).

〔8〕 S. I. Pai, *Magnetogasdynamics and Plasmadynamics, Prentice-Hall/Springer* Verlag (1962).

〔9〕 E. L. Resler, Jr., and J. E. McCune, *Some Exact Solutions in Linearized Magnetoaerodynamics for Arbitrary Magnetic Reynolds Numbers, Rev. Modern Phys.* **32**, 848 (1960).

〔10〕 W. R. Sears, *Sub-Alfvenic Flow in Magnetoaerodynamics, J. Aero/Space Sci.* **28**, 249 (1961).

〔11〕 W. R. Sears, and E. L., Resler, Jr., *Theory of Thin Airfoils in Fluids of High Electrical Conductivity, J. Fluid Mech.* **5**, 257 (1959).

〔12〕 W. R. Sears, and E. L., Resler, Jr., *Sub- and Super-Alfvenic Flows Past Bodies, "Advances in Aeronautical Sciences", Pergamon Press*, **4**, 657 (1961).

〔13〕 A. R. Seebass, *The Theory of Aligned-Fields Magnetogasdynamic Flows*, Cornell University, Ithaca, New York (1962).

〔14〕 K. S. Sung, *Magnetogasdynamic Flow Past Thin Bodies for Arbitrary Mach Number, Alfven Number and Magnetic Reynolds Number, Ph. D. Thesis*, Harvard University. (1963).

(15) T. Taniuti, *An Example of Isentropic Steady Flow in Magnetohydrodynamics. Progress Theor. Phys.* **19**, 749 (1958).

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